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1980 J. Phys. A: Math. Gen. 13 3657

(http://iopscience.iop.org/0305-4470/13/12/015)

J. Phys. A: Math. Gen. 13 (1980) 3657–3670. Printed in Great Britain

An identification of the metric tensor of the non-symmetric unified field theory

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Received 22 August 1979, in final form 5 March 1980

Abstract. A description of a non-Riemannian geometry is given in which use is made of a pair of affine connections to characterise the manifold. This approach is used to discuss the 'Palatini' variational principles of general relativity and the non-symmetric unified theory. In the latter case, an identification of the metric tensor is generated by the variational principle. The implications of the adoption of this identification are discussed.

1. Introduction

Throughout its history, the well-known non-symmetric unified field theory has encountered many difficulties, the great majority of which have arisen simply because of the uncertainty which attends the connection of the formalism of the theory with physics. That is to say, the concepts upon which the theory is founded appear to give no indications as to which of the tensors used in its construction correspond to the physical electromagnetic and gravitational fields.

In the electromagnetic case, the absence of any analogue of the principle of equivalence implies that we have no way, *a priori*, to relate this field to any particular geometrical quantity. This appears to necessitate additional postulates—exterior to the basic concepts of the theory—which enable us to relate the mathematics to our assumption that the electromagnetic field should be associated with an antisymmetric second-rank tensor. In short, we have no choice but to postulate that some quantity within the theory shall play the role of the electromagnetic field tensor. Our only guides in making this otherwise arbitrary choice are our expectations as to the form that the description of electromagnetism should take. For example, we anticipate general field laws which resemble, in some manner, the Maxwell or Born–Infeld equations. But in that these expectations are based on a range of experimental data which is, according to the theory itself, by no means exhaustive, this is plainly a somewhat dubious procedure. In the electromagnetic case, unfortunately, no alternative suggests itself. But this is not true of the gravitational case.

The situation of gravitation is quite different. Here we have, *a priori*, good reason to believe that gravitation is an expression of the variations of the metric of space–time (see § 2 below). This definite, pre-assigned geometrical role for the gravitational field leads us to hope that in this case it will not be necessary to postulate an 'identification'. It is quite possible to discuss non-Riemannian geometry in general without assuming any particular relationship whatever between the metric and the affine connection. Thus, it is possible to construct a geometrical variational principle, involving the metric, before the dependence of the latter on the connection of the manifold is known. We may then try to use the variational principle to derive the form of this dependence. This, indeed, is precisely what is done in the case of the application of the well-known Palatini method of variation to general relativity. In the non-symmetric theory, where the fundamental tensor $g_{\mu\nu} \neq g_{\nu\mu}$ and the metric tensor $a_{\mu\nu} = a_{\nu\mu}$ are not assumed to coincide, it is usual to use the Palatini technique only to relate $g_{\mu\nu}$ to the affine connection, $\Gamma^{\alpha}_{\mu\nu}$. In the present work we shall show that this technique may also be used to relate $a_{\mu\nu}$ to $\Gamma^{\alpha}_{\mu\nu}$. Our attempt is therefore no more than a logical extension of the Palatini approach. Furthermore, it enables us to avoid the situation, discussed earlier, which obtains in the electromagnetic case. Lastly, there is of course an aesthetic appeal in a method which enables us to minimise our hypotheses.

2. Non-Riemannian geometry

From a geometrical point of view, the formalism of general relativity may be extended in two directions. Firstly, we can admit torsion (that is, allow the connection to be non-symmetric); secondly, we may allow the covariant derivative of the metric tensor to differ from zero. Wider generalisations are possible, but unnecessary for our purposes.

This broad non-Riemannian framework is not of value in general relativity, for the following reason. If we assume, as is natural, that the absence of a field corresponds to a Minkowskian geometry, it follows that the geometry of space–time is locally Minkowskian as viewed from a 'freely falling' frame of reference. In that frame (and therefore in every frame), the torsion is zero and the metric is a covariant constant. As this is so at any point, we are brought back to the Riemannian case. Thus we see that, essentially because of the principle of equivalence, gravitation is indeed specifically related to the metric tensor.

Let $a_{\mu\nu}$ be the metric tensor, and V^{μ} an arbitrary vector. Upon parallel transport,

$$\delta V^{\mu} = -V^{\alpha}\delta x^{\beta}\Gamma^{\mu}_{\beta\alpha}, \tag{1}$$

and one may easily verify that if ∇_{α} is the corresponding covariant derivative operator, the squared length of V^{μ} changes according to

$$\delta(a_{\mu\nu}V^{\mu}V^{\nu}) = (\nabla_{\alpha}a_{\mu\nu})\delta x^{\alpha}V^{\mu}V^{\nu}. \tag{2}$$

Now let us consider the parallel transport of an arbitrary covariant vector U_{μ}. The equation

$$\delta U_{\mu} = +\delta x^{\alpha}U_{\beta}\Gamma^{\beta}_{\alpha\mu} \tag{3}$$

is customarily derived from (1) by means of a demand that scalars should be invariant upon parallel transport. But of course lengths are scalars; yet, according to (2), they are not preserved upon transport. It is therefore not reasonable to maintain (3) in the context of these general geometries.

A more logical procedure is as follows. We shall replace (1) and (3) by

$$\delta V^{\mu} = -\delta x^{\alpha}V^{\beta}\overset{+}{\Gamma}{}^{\mu}_{\alpha\beta} \qquad \delta U_{\mu} = +\delta x^{\alpha}U_{\beta}\overset{-}{\Gamma}{}^{\beta}_{\alpha\mu} \tag{4}$$

where, in general, $\overset{+}{\Gamma}{}^{\alpha}_{\mu\nu}$ and $\overset{-}{\Gamma}{}^{\alpha}_{\mu\nu}$ are two different affine connections. (The + and − notation should not be confused with Einstein's (1954) conventions, of which we make no use here.) Then, for the scalar $U_\mu V^\mu$ we find

$$\delta(U_\mu V^\mu) = W_\alpha{}^\mu{}_\rho \delta x^\alpha U_\mu V^\rho \tag{5}$$

where

$$W_\alpha{}^\mu{}_\rho = \overset{-}{\Gamma}{}_\alpha{}^\mu{}_\rho - \overset{+}{\Gamma}{}_\alpha{}^\mu{}_\rho . \tag{6}$$

This quantity is a tensor. Upon parallel transport around an infinitesimal circuit characterised by the antisymmetric tensor $A^{\alpha\beta}$, V^μ and U_μ undergo changes

$$\Delta V^\mu = \overset{+}{R}_{\beta\alpha\rho}{}^\mu V^\rho A^{\alpha\beta} \qquad \Delta U_\mu = -\overset{-}{R}_{\beta\alpha\mu}{}^\rho U_\rho A^{\alpha\beta}, \tag{7}$$

where†

$$\overset{+}{R}_{\beta\alpha\mu}{}^\rho = 2\,\partial_{[\beta}\overset{+}{\Gamma}{}_{\alpha]}{}^\rho{}_\mu + 2\overset{+}{\Gamma}{}^\rho_{[\beta|\epsilon|}\overset{+}{\Gamma}{}^\epsilon_{\alpha]\mu} \tag{8}$$

and similarly for $\overset{-}{R}_{\beta\alpha\mu}{}^\rho$. Then also

$$\Delta(U_\mu V^\mu) = (\overset{+}{R}_{\beta\alpha\mu}{}^\rho - \overset{-}{R}_{\beta\alpha\mu}{}^\rho) V^\mu U_\rho A^{\alpha\beta}.$$

The bracketed expression evidently plays the role of a 'curvature' for scalars.

If a metric is now introduced, equation (2) may be generalised, for a pair of arbitrary vectors U_μ, V^μ, to

$$\delta(a_{\mu\nu}U^\mu V^\nu) = (\overset{+}{\nabla}_\alpha a_{\mu\nu})\delta x^\alpha U^\mu V^\nu \tag{9}$$

where $\overset{+}{\nabla}_\alpha$ corresponds to $\overset{+}{\Gamma}{}^\alpha_{\mu\nu}$. The latter is used, of course, because U^μ and V^ν are contravariant vectors. Using $a_{\mu\nu}$ and its inverse $a^{\mu\nu}$ to lower and raise indices, we have also

$$\delta(a^{\mu\nu}U_\mu V_\nu) = (\overset{-}{\nabla}_\alpha a^{\mu\nu})\delta x^\alpha U_\mu V_\nu . \tag{10}$$

Comparison of (9) and (10) with (5) yields

$$\overset{+}{\nabla}_\alpha a_{\mu\nu} = W_{\alpha\mu\nu} = a_{\mu\lambda} W_{\alpha\cdot\nu}^{\ \lambda} \qquad \overset{-}{\nabla}_\alpha a^{\mu\nu} = W_\alpha{}^{\mu\nu} = a^{\nu\lambda} W_{\alpha\cdot\lambda}^{\ \mu}. \tag{11}$$

Assuming the covariant constancy of δ^λ_ρ, we obtain

$$\overset{+}{\nabla}_\alpha a^{\mu\nu} = -W_\alpha{}^{\mu\nu} \qquad \overset{-}{\nabla}_\alpha a_{\mu\nu} = -W_{\alpha\mu\nu}. \tag{12}$$

Letting a be the determinant of $a_{\mu\nu}$,

$$\overset{+}{\nabla}_\alpha \sqrt{-a} = \tfrac{1}{2}\sqrt{-a}\, W_\alpha{}^\mu{}_\mu \qquad \overset{-}{\nabla}_\alpha \sqrt{-a} = -\tfrac{1}{2}\sqrt{-a}\, W_\alpha{}^\mu{}_\mu . \tag{13}$$

Let us now expand the covariant derivative operator in the first equation (11), and substitute for $W_\alpha{}^\lambda{}_\nu$ from (6). Then

$$\partial_\alpha a_{\mu\nu} - a_{\lambda\nu}\overset{+}{\Gamma}{}^\lambda_{\alpha\mu} - a_{\mu\lambda}\overset{+}{\Gamma}{}^\lambda_{\alpha\nu} = a_{\mu\lambda}(\overset{-}{\Gamma}{}^\lambda_{\alpha\nu} - \overset{+}{\Gamma}{}^\lambda_{\alpha\nu})$$

or

$$\partial_\alpha a_{\mu\nu} - a_{\lambda\nu}\overset{+}{\Gamma}{}^\lambda_{\alpha\mu} - a_{\mu\lambda}\overset{-}{\Gamma}{}^\lambda_{\alpha\nu} = 0. \tag{14}$$

It must be clearly understood that we have made no hypotheses or specialisations whatever in arriving at (14). The fact that $\overset{+}{\Gamma}{}^\lambda_{\alpha\mu}$ and $\overset{-}{\Gamma}{}^\lambda_{\alpha\mu}$ differ (which is bound up with the fact that lengths are distorted under the action of parallel transport) is an expression of an essentially metric property of the manifold. From the mere definitions of $\overset{+}{\Gamma}{}^\alpha_{\mu\nu}$ and $\overset{-}{\Gamma}{}^\alpha_{\mu\nu}$, together with (2), it must follow that given these two connections, the covariant

† Square brackets denote the antisymmetric part.

derivative of $a_{\mu\nu}$ is essentially determinate. This is expressed by (14). We repeat that (14) is a consequence of our definitions, and not of any assumption whatever. It must hold in every geometry. If, of course, $\overset{+}{\Gamma}{}^{\alpha}_{\mu\nu}$ and $\overset{-}{\Gamma}{}^{\alpha}_{\mu\nu}$ are equal, then according to (6) $W_{\alpha}{}^{\mu}{}_{\rho}$ is zero. This agrees with (14), for in that case this latter is

$$\overset{+}{\nabla}_{\alpha} a_{\mu\nu} = 0$$

and according to (11) this again means that $W_{\alpha\mu\nu}$ is zero. In other words, $W_{\alpha\mu\nu}$ is zero if and only if $a_{\mu\nu}$ is a covariant constant.

3. Application: the Palatini method in general relativity

In this section, we reformulate the general relativistic 'Palatini method of variation' in terms of the approach outlined in the preceding section. In a general non-Riemannian geometry, the metric and the affine connections are independently defined and independently meaningful quantities. However, the variational principle of general relativity is constructed from a mixture of these; if, therefore, it is to constitute an intelligible restriction on possible gravitational fields, there must be found some definite relationship between the metric and the affine quantities. This relationship may be derived from the variational principle itself by permitting independent variations of $a_{\mu\nu}$ and $\Gamma^{\alpha}_{\mu\nu}$. The initial geometry is assumed to be torsionless but not Riemannian.

In seeking to generalise the usual equation

$$\delta \int R\sqrt{-a}\, \mathrm{d}\Omega = 0,$$

where R is the curvature scalar, we shall be guided by the realisation that there can be no justification whatever for giving preference to one of $\overset{+}{\Gamma}{}^{\alpha}_{\mu\nu}$, $\overset{-}{\Gamma}{}^{\alpha}_{\mu\nu}$ at the expense of the other. Therefore, we shall take

$$\delta \int (\overset{+}{R}_{\alpha\beta} + \overset{-}{R}_{\alpha\beta}) a^{\alpha\beta} \sqrt{-a}\, \mathrm{d}\Omega = 0. \tag{15}$$

Great caution must be exercised here as to the choice of variational parameters. It is clear, from (14), that it will not be possible to vary all three of $a_{\mu\nu}$, $\overset{+}{\Gamma}{}^{\alpha}_{\mu\nu}$ and $\overset{-}{\Gamma}{}^{\alpha}_{\mu\nu}$ independently. We can only permit the independent variation of either of the pairs $(a_{\mu\nu}, \overset{+}{\Gamma}{}^{\alpha}_{\mu\nu})$ or $(a_{\mu\nu}, \overset{-}{\Gamma}{}^{\alpha}_{\mu\nu})$. It clearly does not matter which pair is chosen; we arbitrarily take the first.

We begin by varying $\overset{+}{\Gamma}{}^{\alpha}_{\mu\nu}$ while keeping $a_{\mu\nu}$ fixed. Then (14) enables us to calculate the variation suffered by $\overset{-}{\Gamma}{}^{\alpha}_{\mu\nu}$. For, by (14) (keeping $a_{\mu\nu}$ fixed),

$$-a_{\rho\nu}\delta\overset{+}{\Gamma}{}^{\rho}_{\alpha\mu} - a_{\mu\rho}\delta\overset{-}{\Gamma}{}^{\rho}_{\alpha\nu} = 0$$

whence

$$\delta\overset{-}{\Gamma}{}^{\rho}_{\alpha\nu} = -a^{\rho\lambda} a_{\nu\sigma}\delta\overset{+}{\Gamma}{}^{\sigma}_{\alpha\lambda}. \tag{16}$$

Let $\Gamma^{\alpha}_{\mu\nu}$ be any connection, and $R_{\mu\lambda}$ the corresponding Ricci tensor. Then we have the well-known formula

$$\delta R_{\mu\lambda} = 2\nabla_{[\mu}\delta\Gamma^{\alpha}_{\alpha]\lambda} + 2\Gamma^{\epsilon}_{[\mu\alpha]}\delta\Gamma^{\alpha}_{\epsilon\lambda}.$$

In the present section, we shall take all connections to be symmetric. Thus

$$\delta \overset{+}{R}_{\mu\lambda} = 2\overset{+}{\nabla}_{[\mu}\delta\overset{+}{\Gamma}{}^{\alpha}_{\alpha]\lambda}.$$

Denoting

$$\mathscr{A}^{\mu\nu} = \sqrt{-a}\, a^{\mu\nu},$$

we have from (15)

$$\int (\overset{+}{\nabla}_\beta \delta\overset{+}{\Gamma}_\alpha{}^\alpha{}_\lambda - \overset{+}{\nabla}_\alpha\delta\overset{+}{\Gamma}{}^\alpha_{\beta\lambda} + \overset{-}{\nabla}_\beta\delta\overset{-}{\Gamma}_\alpha{}^\alpha{}_\lambda - \overset{-}{\nabla}_\alpha\delta\overset{-}{\Gamma}{}^\alpha_{\beta\lambda})\mathscr{A}^{\beta\lambda}\,\mathrm{d}\Omega = 0.$$

The variations are assumed to vanish on the surface of the region of integration, and this leads in the usual way to

$$\int \delta\overset{+}{\Gamma}{}^\alpha_{\beta\lambda}(\overset{+}{\nabla}_\alpha\mathscr{A}^{\beta\lambda} - \delta^\beta_\alpha\overset{+}{\nabla}_\sigma\mathscr{A}^{\sigma\lambda}) + \delta\overset{-}{\Gamma}{}^\alpha_{\beta\lambda}(\overset{-}{\nabla}_\alpha\mathscr{A}^{\beta\lambda} - \delta^\beta_\alpha\overset{-}{\nabla}_\sigma\mathscr{A}^{\sigma\lambda})\,\mathrm{d}\Omega = 0.$$

Using (16) to eliminate $\delta\overset{-}{\Gamma}{}^\alpha_{\beta\lambda}$, we obtain (keeping in mind the symmetry of $\delta\overset{+}{\Gamma}{}^\alpha_{\beta\lambda}$)

$$\overset{+}{\nabla}_\alpha{}^{\beta\lambda} - \tfrac12(\delta^\beta_\alpha\overset{+}{\nabla}_\sigma\mathscr{A}^{\sigma\lambda} + \delta^\lambda_\alpha\overset{+}{\nabla}_\sigma\mathscr{A}^{\sigma\beta}) - \tfrac12(a^{\mu\lambda}a_{\nu\alpha}\overset{-}{\nabla}_\mu\mathscr{A}^{\beta\nu} + a^{\mu\beta}a_{\nu\alpha}\overset{-}{\nabla}_\mu\mathscr{A}^{\lambda\nu})$$
$$+\tfrac12(a^{\mu\lambda}a_{\nu\alpha}\delta^\beta_\mu\overset{-}{\nabla}_\sigma\mathscr{A}^{\sigma\lambda} + a^{\mu\beta}a_{\nu\alpha}\delta^\lambda_\mu\overset{-}{\nabla}_\sigma\mathscr{A}^{\sigma\beta}) = 0. \tag{17}$$

Using (12) and (13), one may show

$$\overset{+}{\nabla}_\alpha\mathscr{A}^{\beta\lambda} = \sqrt{-a}\,(\tfrac12 a^{\beta\lambda}W_\alpha{}^\mu{}_\mu - W_\alpha{}^{\beta\lambda}) \qquad \overset{-}{\nabla}_\alpha\mathscr{A}^{\beta\lambda} = -\sqrt{-a}\,(\tfrac12 a^{\beta\lambda}W_\alpha{}^\mu{}_\mu - W_\alpha{}^{\beta\lambda}).$$

Then

$$\overset{+}{\nabla}_a\mathscr{A}^{\alpha\lambda} = \sqrt{-a}\,(\tfrac12 W^{\delta\mu}{}_\mu - W_\mu{}^{\mu\delta}). \tag{18}$$

As both $\overset{+}{\Gamma}{}^\alpha_{\mu\nu}$ and $\overset{-}{\Gamma}{}^\alpha_{\mu\nu}$ are symmetric, it follows from (6) that

$$W_\mu{}^\alpha{}_\nu = W_\nu{}^\alpha{}_\mu. \tag{19}$$

Hence (18) is

$$\overset{+}{\nabla}_\alpha\mathscr{A}^{\alpha\lambda} = -\tfrac12\sqrt{-a}\;W_\mu{}^{\mu\lambda}.$$

Applying these results, including (19), to (17), one finds after some calculation that

$$\tfrac12\delta^\beta_\alpha W_\sigma{}^{\sigma\lambda} + \tfrac12\delta^\lambda_\alpha W_\sigma{}^{\sigma\beta} + a^{\beta\lambda}W_\sigma{}^\sigma{}_\alpha - 2W_\alpha{}^{\beta\lambda} = 0. \tag{20}$$

Contracting on α and β, we obtain

$$\tfrac32 W_\sigma{}^{\sigma\lambda} = 0.$$

Substituting this into (20), we have

$$W_\alpha{}^\beta{}_\delta = 0. \tag{21}$$

By (6), this means that our two connections coincide; according to (11), the metric is a covariant constant. Thus, if $\Gamma^\alpha_{\mu\nu}$ is the common value of $\overset{+}{\Gamma}{}^\alpha_{\mu\nu}$ and $\overset{-}{\Gamma}{}^\alpha_{\mu\nu}$,

$$\partial_\alpha a_{\mu\nu} - a_{\rho\nu}\Gamma^\rho_{\alpha\mu} - a_{\mu\rho}\Gamma^\rho_{\alpha\nu} = 0,$$

and this establishes the desired relationship between $a_{\mu\nu}$ and $\Gamma^\alpha_{\mu\nu}$. The geometry is of course Riemannian. The variation of $a_{\mu\nu}$ now yields the standard field equations of general relativity. Evidently, then, the Palatini method is not upset when we make the generalisations necessitated by the 'two-connection' approach.

It may be felt that the adoption of a variational principle which is symmetrical in $\overset{+}{\Gamma}{}^{\alpha}_{\mu\nu}$ and $\overset{-}{\Gamma}{}^{\alpha}_{\mu\nu}$ will inevitably lead to (21). But were this comment valid, it would apply also to the case in which the connections are not symmetric. In that case, however, it may be shown that (20) is replaced by

$$\tfrac{1}{2}\delta^{\beta}_{\alpha}W_{\sigma}{}^{\sigma\lambda}+\tfrac{1}{2}\delta^{\lambda}_{\alpha}W_{\sigma}{}^{\sigma\beta}+a^{\beta\lambda}W_{\sigma}{}^{\sigma}{}_{\alpha}-W_{\alpha}{}^{\beta\lambda}-W^{\lambda\beta}{}_{\alpha}=0.$$

As we no longer have (19), we cannot deduce (21) from this. Generally speaking, the symmetry of the Lagrangian would lead us to anticipate that the two connections will satisfy the same equations, but it is frequently far from obvious that these equations cannot have two distinct solutions. (For example, given $a_{\mu\nu}$, there is no unique connection satisfying

$$\partial_{\alpha}a_{\mu\nu}-a_{\rho\nu}\Gamma^{\rho}_{\alpha\mu}-a_{\mu\rho}\Gamma^{\rho}_{\alpha\nu}=0$$

unless $\Gamma^{\alpha}_{\mu\nu}$ is symmetric.)

4. A 'two-connection' approach to the non-symmetric unified field theory

The non-symmetric theory, with the basis of which we assume the reader to be familiar, involves the abandonment of the condition of symmetry imposed on the general-relativistic fundamental tensor and affine connection. The variational principle is a natural extension of the Palatini principle. The variation of the affine connection yields a relationship between it and the fundamental tensor $g_{\mu\nu}\neq g_{\nu\mu}$, which latter bears no immediate relationship to the metric tensor $a_{\mu\nu}$. In the standard account, the variational principle does not provide a means whereby the metric tensor may be related to the affine connection.

Klotz (1978b) has postulated that the metric tensor should be calculated from the equation†

$$\partial_{\alpha}a_{\mu\nu}-a_{\rho\nu}\tilde{\Gamma}^{\rho}_{(\alpha\mu)}-a_{\mu\rho}\tilde{\Gamma}^{\rho}_{(\alpha\nu)}=0, \tag{22}$$

where $\tilde{\Gamma}^{\alpha}_{\mu\nu}$ is discussed below. It is of central importance that this 'identification' of the metric tensor is not compatible with every solution of the field equations of the theory. Like the latter, then, it constitutes a restriction on the range of geometries available to space–time, and is itself akin to a field equation. The possibility that the actual metric identification may be of this type clearly adds weight to our claim that it should, like the other field equations, be derived from the variational principle.

For the latter we take

$$\delta\int(\overset{+}{R}_{\mu\nu}+\overset{-}{R}_{\mu\nu})g^{\mu\nu}\sqrt{-g}\,\mathrm{d}\Omega=0, \tag{23}$$

as the natural generalisation of (15). Here, because $g_{\mu\nu}$, $\overset{+}{\Gamma}{}^{\alpha}_{\mu\nu}$ and $\overset{-}{\Gamma}{}^{\alpha}_{\mu\nu}$ are not related *a priori* by an equation of the form (14), it becomes possible to vary all three sets of quantities independently. Thus

$$\int(\delta\overset{+}{R}_{\mu\nu})g^{\mu\nu}\sqrt{-g}\,\mathrm{d}\Omega=0 \qquad \int(\delta\overset{-}{R}_{\mu\nu})g^{\mu\nu}\sqrt{-g}\,\mathrm{d}\Omega=0$$

$$\int(\overset{+}{R}_{\mu\nu}+\overset{-}{R}_{\mu\nu})\delta(g^{\mu\nu}\sqrt{-g})\,\mathrm{d}\Omega=0.$$

† Round brackets denote the symmetric part.

(The fact that there is no *a priori* relationship between $\overset{+}{\Gamma}{}^{\alpha}_{\mu\nu}$, $\bar{\Gamma}{}^{\alpha}_{\mu\nu}$ and $g_{\mu\nu}$ is a result of the fact that $g_{\mu\nu}$ is not endowed with any direct geometrical meaning. Its purpose is to restrict the excessive number of 'degrees of freedom' which the general geometry possesses. See Klotz (1978b).)

The standard procedures may now be applied to the above equations (see Schrödinger 1950). If we define

$$\overset{+}{\tilde{\Gamma}}{}^{\alpha}_{\mu\nu}=\overset{+}{\Gamma}{}^{\alpha}_{\mu\nu}+\tfrac{2}{3}\overset{+}{\Gamma}_{\mu}\delta^{\alpha}_{\nu} \qquad \bar{\tilde{\Gamma}}{}^{\alpha}_{\mu\nu}=\bar{\Gamma}{}^{\alpha}_{\mu\nu}+\tfrac{2}{3}\bar{\Gamma}_{\mu}\delta^{\alpha}_{\nu} \tag{24}$$

where

$$\overset{+}{\Gamma}_{\mu}=\overset{+}{\Gamma}{}^{\alpha}_{[\alpha\mu]} \qquad \bar{\Gamma}_{\mu}=\bar{\Gamma}{}^{\alpha}_{[\alpha\mu]},$$

then we obtain

$$\partial_{\alpha}g_{\mu\nu}-g_{\rho\nu}\overset{+}{\tilde{\Gamma}}{}^{\rho}_{\alpha\mu}-g_{\mu\rho}\overset{+}{\tilde{\Gamma}}{}^{\rho}_{\nu\alpha}=0 \qquad \partial_{\alpha}g_{\mu\nu}-g_{\rho\nu}\bar{\tilde{\Gamma}}{}^{\rho}_{\alpha\mu}-g_{\mu\rho}\bar{\tilde{\Gamma}}{}^{\rho}_{\nu\alpha}=0 \tag{25}$$

and also

$$\begin{gathered}\overset{+}{\tilde{\boldsymbol{R}}}_{(\mu\nu)}+\bar{\tilde{\boldsymbol{R}}}_{(\mu\nu)}=0\\ \partial_{\lambda}(\overset{+}{\tilde{\boldsymbol{R}}}_{[\mu\nu]}+\bar{\tilde{\boldsymbol{R}}}_{[\mu\nu]})+\partial_{\mu}(\overset{+}{\tilde{\boldsymbol{R}}}_{[\nu\lambda]}+\bar{\tilde{\boldsymbol{R}}}_{[\nu\lambda]})+\partial_{\nu}(\overset{+}{\tilde{\boldsymbol{R}}}_{[\lambda\mu]}+\bar{\tilde{\boldsymbol{R}}}_{[\lambda\mu]})=0\\ \overset{+}{\tilde{\Gamma}}_{\mu}=\bar{\tilde{\Gamma}}_{\mu}=0.\end{gathered} \tag{26}$$

Here $\overset{\pm}{\tilde{\boldsymbol{R}}}_{\mu\nu}$ is the Ricci tensor formed from $\overset{\pm}{\tilde{\Gamma}}{}^{\alpha}_{\mu\nu}$, and so on.

Now Tonnelat (1955) has shown that, in general, equations of the form (25) have a unique solution for the affine connection in terms of the fundamental tensor $g_{\mu\nu}$. Thus it follows from (25) that

$$\overset{+}{\tilde{\Gamma}}{}^{\alpha}_{\mu\nu}=\bar{\tilde{\Gamma}}{}^{\alpha}_{\mu\nu}(=\tilde{\Gamma}^{\alpha}_{\mu\nu}), \tag{27}$$

and then the field equations become

$$\begin{aligned}&\partial_{\alpha}g_{\mu\nu}-g_{\rho\nu}\tilde{\Gamma}^{\rho}_{\alpha\mu}-g_{\mu\rho}\tilde{\Gamma}^{\rho}_{\nu\alpha}=0 &&\tilde{\boldsymbol{R}}_{(\mu\nu)}=0\\ &\partial_{\lambda}\tilde{\boldsymbol{R}}_{[\mu\nu]}+\partial_{\mu}\tilde{\boldsymbol{R}}_{[\nu\lambda]}+\partial_{\nu}\tilde{\boldsymbol{R}}_{[\lambda\mu]}=0 &&\tilde{\Gamma}_{\mu}=0.\end{aligned} \tag{28}$$

This is the standard set. We have therefore demonstrated that, as in the case of general relativity, the variational principle of the non-symmetric theory may be adapted to the requirements of the 'two-connection' approach, without affecting the character of the resulting theory. But in addition to this, we have the crucial result (27). If this is substituted into (14) (which, as we know, holds quite generally), there results

$$\partial_{\alpha}a_{\mu\nu}-a_{\rho\nu}\tilde{\Gamma}^{\rho}_{\alpha\mu}-a_{\mu\rho}\tilde{\Gamma}^{\rho}_{\alpha\nu}=0. \tag{29}$$

This we now regard as a set of differential equations for the metric tensor. In other words, (29) is our metric identification. It arises from (25) through (27); evidently, then, we have achieved our aim of using the variational principle to relate not only $g_{\mu\nu}$, but also $a_{\mu\nu}$, to $\Gamma^{\alpha}_{\mu\nu}$.

Many other solutions of the problem of 'identifying' the metric (that is, of establishing a relationship between the metric and affine properties of the relevant manifold) have been proposed (see, for example, Wyman (1950) and Schrödinger (1947)). Of particular interest, because of its close similarity to (29), is the already mentioned suggestion (22) due to Klotz (1978b). After demonstrating the integrability of (22), Klotz shows that, in the case of Papapetrou's (1948) spherically symmetric static fundamental tensor, it eliminates all but two of the corresponding solutions due to

Vanstone (1962). One of these, according to the identification of $\tilde{R}_{[\mu\lambda]}$ as the electromagnetic field (due to Gregory and Klotz (1977)), is a spherically symmetric (Coulomb) electric field; the other is a spherically symmetric magnetic solution.

In the sections which follow, we perform the analogous calculations for the condition (29).

5. The integrability of the metric condition

In this section we generalise the proof of integrability given by Klotz for (22). We begin by establishing an identity for the curvature $\tilde{R}_{\mu\nu\alpha}{}^{\beta}$ corresponding to a $\tilde{\Gamma}^{\alpha}_{\mu\nu}$ which satisfies (29). If $\tilde{\nabla}_\alpha$ is the corresponding covariant derivative operator, and V_μ any vector, we have the well-known result

$$\tilde{\nabla}_{[\alpha}\tilde{\nabla}_{\beta]}V_\mu = -\tfrac{1}{2}\tilde{R}_{\alpha\beta\mu}{}^{\rho}V_\rho - \tilde{\Gamma}^{\rho}_{[\alpha\beta]}\tilde{\nabla}_\rho V_\mu.$$

Applying this to the metric tensor $a_{\mu\nu}$, one finds

$$\tilde{\nabla}_{[\alpha}\tilde{\nabla}_{\beta]}a_{\mu\nu} = -\tfrac{1}{2}\tilde{R}_{\alpha\beta\mu}{}^{\rho}a_{\rho\nu} - \tfrac{1}{2}\tilde{R}_{\alpha\beta\nu}{}^{\rho}a_{\mu\rho} - \tilde{\Gamma}^{\rho}_{[\alpha\beta]}\tilde{\nabla}_\rho a_{\mu\nu}. \tag{30}$$

But (29) states

$$\tilde{\nabla}_\alpha a_{\mu\nu} = 0,$$

so that we have from (30)

$$0 = \tilde{R}_{\alpha\beta\mu\nu} + \tilde{R}_{\alpha\beta\nu\mu}. \tag{31}$$

Now by differentiating (29) one may easily show that

$$\partial_\beta\,\partial_\alpha a_{\mu\nu} = a_{\lambda\nu}\tilde{\Gamma}^{\lambda}_{\beta\rho}\tilde{\Gamma}^{\rho}_{\alpha\mu} + a_{\rho\lambda}\tilde{\Gamma}^{\lambda}_{\beta\nu}\tilde{\Gamma}^{\rho}_{\alpha\mu} + a_{\rho\nu}\,\partial_\beta\tilde{\Gamma}^{\rho}_{\alpha\mu} + a_{\lambda\rho}\tilde{\Gamma}^{\lambda}_{\beta\mu}\tilde{\Gamma}^{\rho}_{\alpha\nu} + a_{\mu\lambda}\tilde{\Gamma}^{\lambda}_{\beta\rho}\tilde{\Gamma}^{\rho}_{\alpha\nu} + a_{\mu\rho}\,\partial_\beta\tilde{\Gamma}^{\rho}_{\alpha\nu}.$$

Observing that the sum of the second and fourth terms is symmetric in α and β, one sees that

$$\partial_{[\beta}\,\partial_{\alpha]}a_{\mu\nu} = a_{\rho\nu}\tilde{R}_{\beta\alpha\mu}{}^{\rho} + a_{\mu\rho}\tilde{R}_{\beta\alpha\nu}{}^{\rho}$$

or

$$\partial_{[\beta}\partial_{\alpha]}a_{\mu\nu} = \tilde{R}_{\beta\alpha\mu\nu} + \tilde{R}_{\beta\alpha\nu\mu}.$$

By (31), then, the integrability condition

$$\partial_{[\beta}\partial_{\alpha]}a_{\mu\nu} = 0$$

is assured. The situation is precisely the same as for the Klotz condition (22).

6. The spherically symmetric solutions

As has already been mentioned, neither (22) nor (29) is compatible with every solution of the usual field equations. In the former case, Klotz (1978b) has calculated, for the spherically symmetric static solutions, the conditions which must be satisfied if a metric is to be generated. Here we shall modify these calculations so as to deal with (29).

Papapetrou's spherically symmetric, time-independent fundamental tensor is

$$g_{11} = -\alpha \qquad g_{22} = -\beta = g_{33}\operatorname{cosec}^2\theta$$

$$g_{44} = \sigma \qquad g_{23} = -g_{32} = f\sin\theta \qquad g_{14} = -g_{41} = \omega \tag{32}$$

where α, β, σ, f and ω depend on $x^1 = r$ only. Define

$$U = 1 - \omega^2/\alpha\sigma \qquad \rho^2 = f^2 + \beta^2 \qquad y = \sigma U$$
$$x = \rho^2/\alpha \qquad A' = \rho'/\rho \qquad \tan B = \beta/f \tag{33}$$

where the prime denotes differentiation with respect to r. Tonnelat (1955) has shown that the non-zero components of $\tilde{\Gamma}^{\alpha}_{\mu\nu}$ are for this case given by

$$\begin{aligned}
&\tilde{\Gamma}^{1}_{11} = \alpha'/2\alpha \qquad \tilde{\Gamma}^{1}_{22} = \tilde{\Gamma}^{1}_{33}\operatorname{cosec}^2\theta = (fB' - \beta A')/2\alpha \\
&\tilde{\Gamma}^{1}_{44} = (\sigma/2\alpha)(\ln yU)' \qquad \tilde{\Gamma}^{2}_{33} = -\sin\theta\cos\theta \\
&\tilde{\Gamma}^{3}_{(23)} = \cot\theta \qquad \tilde{\Gamma}^{2}_{(12)} = \tilde{\Gamma}^{3}_{(13)} = \tfrac{1}{2}A' \\
&\tilde{\Gamma}^{2}_{(34)} = -\tilde{\Gamma}^{3}_{(24)}\sin^2\theta = (\omega B'/2\alpha)\sin\theta \\
&\tilde{\Gamma}^{4}_{(14)} = \tfrac{1}{2}y'/y \qquad \tilde{\Gamma}^{1}_{[23]} = (1/2\alpha)(\beta B' + fA')\sin\theta \\
&\tilde{\Gamma}^{2}_{[31]} = \tilde{\Gamma}^{3}_{[12]}\sin^2\theta = -\tfrac{1}{2}B'\sin\theta \\
&\tilde{\Gamma}^{1}_{[14]} = -2\tilde{\Gamma}^{2}_{[24]} = -2\tilde{\Gamma}^{3}_{[34]} = \omega\rho'/\alpha\rho.
\end{aligned}\tag{34}$$

Here the coordinate system is $x^{\mu} = (r\theta\phi t)$.

Taking the case for which $a_{\mu\nu}$ is diagonal and dependent only upon r and θ, one may write out the forty equations

$$\partial_\alpha a_{\mu\nu} = a_{\rho\nu}\tilde{\Gamma}^{\rho}_{\alpha\mu} + a_{\mu\rho}\tilde{\Gamma}^{\rho}_{\alpha\nu}$$

in full. In doing this, we observe the following important fact about the surviving components (34): that if a certain lower index pair occurs among the symmetric set, it does not occur among the antisymmetric set and vice versa. (For example, $\tilde{\Gamma}^{3}_{(23)} \neq 0$ but $\tilde{\Gamma}^{3}_{[23]} = 0$, and $\tilde{\Gamma}^{2}_{[31]} \neq 0$ but $\tilde{\Gamma}^{2}_{(31)} = 0$.) This implies that the conditions

$$\begin{aligned}
&a_{11} = a_0\alpha \qquad a_{22} = a_{33}\operatorname{cosec}^2\theta = b_0\rho \\
&a_{44} = y/y_0 \qquad \tilde{\Gamma}^{2}_{(12)}a_{22} + \tilde{\Gamma}^{1}_{22}a_{11} = 0 \\
&\tilde{\Gamma}^{4}_{(14)}a_{44} + \tilde{\Gamma}^{1}_{44}a_{11} = 0 \qquad \tilde{\Gamma}^{3}_{(13)}a_{33} + \tilde{\Gamma}^{1}_{33}a_{11} = 0 \qquad \tilde{\Gamma}^{2}_{(34)} = 0
\end{aligned}\tag{35}$$

derived by Klotz are left intact. These are now supplemented, however, by a set resulting from the inclusion of the antisymmetric part of the connection:

$$\tilde{\Gamma}^{1}_{[41]} = \tilde{\Gamma}^{2}_{[42]} = \tilde{\Gamma}^{3}_{[34]} = 0 \tag{36}$$

$$a_{22}\tilde{\Gamma}^{2}_{[31]} + a_{11}\tilde{\Gamma}^{1}_{[32]} = 0 \tag{37a}$$

$$a_{33}\tilde{\Gamma}^{3}_{[21]} + a_{11}\tilde{\Gamma}^{1}_{[23]} = 0 \tag{37b}$$

$$a_{33}\tilde{\Gamma}^{3}_{[12]} + a_{22}\tilde{\Gamma}^{2}_{[13]} = 0. \tag{37c}$$

From the Tonnelat solution (34), it follows that any one of (36) implies the other two, while it is clear that any two of (37a), (37b), (37c) imply the third. Quoting from (34) and (35), we have

$$\tilde{\Gamma}^{2}_{[13]} = \tilde{\Gamma}^{3}_{[21]}\sin^2\theta \qquad a_{22} = a_{33}\operatorname{cosec}^2\theta.$$

Substitution of these into (37c) shows the latter to be an identity. Our remaining additional conditions are now

$$\tilde{\Gamma}^{1}_{[41]} = 0 \qquad a_{33}\tilde{\Gamma}^{3}_{[21]} = a_{11}\tilde{\Gamma}^{1}_{[32]}. \tag{38}$$

By means of appropriate substitutions from (34) and (35), the second of these may be shown to be

$$(a_0\beta + b_0\rho)B' + a_0 f\rho'/\rho = 0. \tag{39}$$

Following Klotz, we choose, without loss of generality,

$$a_0 = b_0 = -1, \tag{40}$$

so that (39) is

$$(\beta + \rho)B' + f\rho'/\rho = 0. \tag{41}$$

Now Klotz (1978b) has shown that (35) implies (with (40)) that

$$\rho' + fB' - \beta\rho'/\rho = 0 \qquad y' - y_0\sigma(\ln yU)' = 0 \qquad \omega B' = 0. \tag{42}$$

From the first of these,

$$B' = (\rho'/f\rho)(\beta - \rho),$$

whence

$$(\beta + \rho)B' = (\rho'/f\rho)(\beta^2 - \rho^2) = -\rho' f/\rho,$$

because of the definitions (33). But this is precisely (41). The second condition (38) is thus already contained in (42).

From (34) and the first of (38), we have one condition,

$$\omega\rho' = 0$$

beyond (42). If ω is not zero, then ρ' must be; but this conflicts with the field equation (Klotz 1978b, equation (32))

$$(\beta\rho'/2\alpha\rho)' + \tfrac{1}{2}(\ln \alpha y)'(\beta\rho'/2\alpha\rho) - 1 = 0,$$

which holds if ω is not zero. Thus we finally see that, in this case, our metric identification differs from that proposed by Klotz only in that now ω must of necessity be zero.

Among all the solutions of (28) (found by Vanstone (1962)) corresponding to the fundamental tensor (32), Klotz found only two which were compatible with (42). One of these, an electric solution, had ω equal to zero; the other solution, which corresponds to a spherically symmetric magnetic field, did not. With the metric identification (29), then, the non-symmetric theory admits a unique spherically symmetric static solution, which coincides precisely with the Klotz electric solution and the associated cosmology (see Klotz 1978a). There exist no magnetic solutions of this type whatever.

The magnetic solution admitted by (22) involves an inverse fifth power, a fact which complicates its physical interpretation. Nevertheless, it seems clear that the most natural interpretation of this solution is that it corresponds to some kind of 'magnetically charged' particle. It is of course conceivable that such a particle exists: indeed, in the absence of further solutions for which the above process can be carried out, the demonstration of its existence is the only empirical means whereby (22) and (29) may be distinguished. However, it must be admitted that the weight of experimental evidence is against the existence of particles of this type. In the view of the present author, this should be regarded as an advantage of (29) over (22).

7. Transposition invariance

Thus far, we have presented the non-symmetric theory as being founded exclusively on the variational principle (23). Einstein (1954) himself would appear to have been reluctant to proceed in this manner; his primary guide was the principle of 'transposition invariance', which may be described as follows.

The transposition conjugate of a quantity

$$U_{\rho\sigma} = U_{\rho\sigma}(g_{\mu\nu}, \tilde{\Gamma}^{\alpha}_{\mu\nu})$$

is

$$U_{\rho\sigma}{}^{+} = U_{\sigma\rho}(g_{\nu\mu}, \tilde{\Gamma}^{\alpha}_{\mu\nu}).$$

By requiring that all relations should be invariant under such a substitution, Einstein is able suitably to restrict the range of acceptable field equations. The physical motivation for imposing this principle is the expectation that charge conjugation invariance should be reflected in some formal symmetry of the theory. It is clear that this relatively vague physical motivation permits considerable latitude as to the precise manner in which the principle of transposition invariance is to be interpreted. Most importantly, we do not know whether to require transposition invariance with respect to the conjugation of $g_{\mu\nu}$ and $\tilde{\Gamma}^{\alpha}_{\mu\nu}$ only; it would be possible to require it also with respect to the conjugation of other quantities, such as the metric. The former requirement will be referred to as the 'weak' and the latter as the 'strong' principle of transposition invariance.

Let us, for the present, adopt the strong form of the principle. Then, because

$$\partial_{\alpha}a_{\mu\nu} - a_{\rho\nu}\tilde{\Gamma}^{\rho}_{\mu\alpha} - a_{\mu\rho}\tilde{\Gamma}^{\rho}_{\nu\alpha} = 0$$

is not in general compatible with (29) (from which it is derived by transposition conjugation), it follows that our proposed metric identification is not transposition invariant in this sense.

If one places, as did Einstein, more emphasis on transposition invariance as a guide in the selection of field equations than on the variational principle, then one may be willing to step outside the framework of the latter to some extent. Let us, therefore, investigate some immediate modifications of (29) which bring it into accord with the 'strong' principle of transposition invariance. One possibility is to reverse the indices of $\tilde{\Gamma}^{\rho}_{\alpha\nu}$ in the third term of (29), so obtaining

$$\partial_{\alpha}a_{\mu\nu} - a_{\rho\nu}\tilde{\Gamma}^{\rho}_{\alpha\mu} - a_{\mu\rho}\tilde{\Gamma}^{\rho}_{\nu\alpha} = 0. \tag{43}$$

Adding and subtracting a term $a_{\mu\rho}\tilde{\Gamma}^{\rho}_{\alpha\nu}$, and denoting

$$\tilde{S}_{\mu\cdot\nu}^{\ \alpha} = \tilde{\Gamma}^{\alpha}_{[\mu\nu]},$$

we have

$$\tilde{\nabla}_{\alpha}a_{\mu\nu} = 2\tilde{S}_{\nu\mu\alpha} = 2a_{\mu\rho}\tilde{S}_{\nu\cdot\alpha}^{\ \rho},$$

whence clearly

$$\tilde{S}_{[\mu\nu]\alpha} = 0. \tag{44}$$

From the definition,

$$\tilde{S}_{\mu\nu\alpha} = -\tilde{S}_{\alpha\nu\mu}, \tag{45}$$

so that from (44)

$$\tilde{S}_{\mu\nu\alpha} = -\tilde{S}_{\nu\alpha\mu}.$$

Symmetrising now over μ and ν, the left side is (according to (44)) intact, while by (45) the right side becomes zero. Hence $\tilde{S}_{\mu\nu\alpha}$ is zero. The suggestion (43), then, would imply that $\tilde{\Gamma}^{\alpha}_{\mu\nu}$ must always be symmetric, which is not acceptable. (For example, referring back to our discussion of the spherically symmetric case, it is easily seen from (34) that

$$\tilde{\Gamma}^{2}_{[31]} = 0$$

would imply B' to be zero. But (36) still holds and still implies ω to be zero. Klotz (1978b) has shown that, under these conditions, the theory collapses into general relativity. With (43), then, the theory would not possess any spherically symmetric static electric solutions.)

The only other simple modification of (29) which suggests itself (from the point of view of 'strong' transposition invariance) is

$$\partial_\alpha a_{\mu\nu} - a_{\rho\nu}\tilde{\Gamma}^{\rho}_{(\alpha\mu)} - a_{\mu\rho}\tilde{\Gamma}^{\rho}_{(\alpha\nu)} = 0. \tag{22}$$

But this is precisely the Klotz metric identification. The condition (22) may now be characterised as the simplest physically meaningful ('strong') transposition invariant modification of the results of the variational principle. If transposition invariance is placed on a logical par with the variational principle, then in a sense (22) may be said to be derivable from the latter (since this is true of (29)).

The above discussion has been given only in order that (22) may be placed in the context of our efforts to derive the identification of the metric from the variational principle. This particular identification is seen to arise from the imposition of the 'strong' form of transposition invariance. However, this latter appears to rest on a somewhat flimsy motivation. Let us discuss a more cogent basis for this invariance (see Einstein 1954, 1950).

Given any connection $\Gamma^{\alpha}_{\mu\nu}$, and any vector V^{μ}, it is easily shown that both of the formations

$$\partial_\alpha V^{\mu} + V^{\rho}\Gamma^{\mu}_{\alpha\rho} \qquad \partial_\alpha V^{\mu} + V^{\rho}\Gamma^{\mu}_{\rho\alpha}$$

are tensors. On this basis, Einstein (1954) argued that we should systematically treat $\Gamma^{\alpha}_{\mu\nu}$ and $\Gamma^{\alpha}_{\nu\mu}$ on an equivalent footing. This may be treated as a motivation for postulating

$$\partial_\alpha g_{\mu\nu} - g_{\rho\nu}\tilde{\Gamma}^{\rho}_{\alpha\mu} - g_{\mu\rho}\tilde{\Gamma}^{\rho}_{\nu\alpha} = 0. \tag{46}$$

Let now $g_{\mu\nu}{}^{+}$ be the fundamental tensor corresponding to the transposed connection, so that

$$\partial_\alpha g_{\mu\nu}{}^{+} - g_{\rho\mu}{}^{+}\tilde{\Gamma}^{\rho}_{\mu\alpha} - g_{\mu\rho}{}^{+}\tilde{\Gamma}^{\rho}_{\alpha\nu} = 0.$$

Interchanging the dummy indices μ and ν, one finds

$$\partial_\alpha g_{\nu\mu}{}^{+} - g_{\rho\mu}{}^{+}\tilde{\Gamma}^{\rho}_{\nu\alpha} - g_{\nu\rho}{}^{+}\tilde{\Gamma}^{\rho}_{\alpha\mu} = 0.$$

Comparison with (46) leads to

$$g_{\mu\nu}{}^{+} = g_{\nu\mu}.$$

As all the quantities of the theory depend, through (46), on $g_{\mu\nu}$, the logical equivalence of $\Gamma^{\alpha}_{\mu\nu}$ and $\Gamma^{\alpha}_{\nu\mu}$ may now be expressed through the demand of invariance with respect to the conjugation of $g_{\mu\nu}$ only. With this motivation, there is no justification for requiring invariance with respect to the conjugation of the metric tensor. The behaviour of $a_{\mu\nu}$, the metric tensor, under the replacement of $\tilde{\Gamma}^{\alpha}_{\mu\nu}$ by $\tilde{\Gamma}^{\alpha}_{\nu\mu}$ must be calculated from

$$\partial_\alpha a_{\mu\nu}{}^{+} - a_{\rho\nu}{}^{+}\tilde{\Gamma}^{\rho}_{\mu\alpha} - a_{\mu\rho}{}^{+}\tilde{\Gamma}^{\rho}_{\nu\alpha} = 0$$

(see (29)). We cannot simply assume $a_{\mu\nu}{}^{+}$ to be equal to $a_{\nu\mu}$ $(= a_{\mu\nu})$. In short, (29) is not incompatible with the 'weak' form of the principle of transposition invariance.

We may summarise the situation as follows: the metric identifications (22) and (29) are those appropriate, respectively, to the 'strong' and 'weak' interpretations of transposition invariance. Equation (29) follows directly from the variational principle of the theory; this is not strictly true of (22). The latter is subject also to other criticisms. For example, (22) implies that $a_{\mu\nu}$ is not a covariant constant with respect to $\tilde{\Gamma}^{\alpha}_{\mu\nu}$. In accordance with (11) and (6), this means that $\overset{+}{\tilde{\Gamma}}{}^{\alpha}_{\mu\nu}$ and $\overset{=}{\tilde{\Gamma}}{}^{\alpha}_{\mu\nu}$ do not coincide. Which, then, of

$$\partial_\alpha g_{\mu\nu} - g_{\rho\nu}\overset{+}{\tilde{\Gamma}}{}^{\rho}_{\alpha\mu} - g_{\mu\rho}\overset{+}{\tilde{\Gamma}}{}^{\rho}_{\nu\alpha} = 0 \qquad \partial_\alpha g_{\mu\nu} - g_{\rho\nu}\overset{=}{\tilde{\Gamma}}{}^{\rho}_{\alpha\mu} - g_{\mu\rho}\overset{=}{\tilde{\Gamma}}{}^{\rho}_{\nu\alpha} = 0$$

(if either) is to hold? It is far from easy to understand how one is to establish a basis for deciding this question.

Ultimately, of course, the decisive issue is the empirical status of the magnetic solution admitted by (22) but not by (29). The verification of the existence of fields of this type would settle not only the question of the metric identification, but also that of whether the 'strong' or 'weak' interpretations of transposition invariance should be imposed.

8. Conclusions

The expression 'metric identification' is slightly misleading, since it gives one the impression that the 'identification' problem is similar for both gravitation and electromagnetism. We have in fact good reason at the outset to associate gravitation with the metric tensor, and the latter has a meaning which is independent of any other entity. What is needed is a relationship between the metric and the affine properties of the manifold. It is conceivable that the variational principle of the theory may itself provide such a relationship. The present work shows that, with an appropriate view of non-Riemannian geometry, this can indeed be achieved, the result being

$$\partial_\alpha a_{\mu\nu} - a_{\rho\nu}\tilde{\Gamma}^{\rho}_{\alpha\mu} - a_{\mu\rho}\tilde{\Gamma}^{\rho}_{\alpha\nu} = 0. \tag{29}$$

If we impose a 'strong' form of the principle of transposition invariance, (29) must be modified to

$$\partial_\alpha a_{\mu\nu} - a_{\rho\nu}\tilde{\Gamma}^{\rho}_{(\alpha\mu)} - a_{\mu\rho}\tilde{\Gamma}^{\rho}_{(\alpha\nu)} = 0, \tag{22}$$

as proposed previously by Klotz (1978b). In the spherically symmetric case, (22) and (29) admit an electric solution which is identical for both; but (22) also admits a magnetic solution, which is ruled out in the case of (29).

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